q,t-Combinatorics in Algebra, Geometry, and Combinatorics

Perspectives on Catalanimals part I

Nonsymmetric Hall-Littlewood polynomials, LLT series, and the Cauchy kernel

George H. Seelinger

based on joint work with

Jonah Blasiak

Mark Haiman

Jennifer Morse

Anna Pun

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Big Picture

Want to show

Algebraic:

Operator applied to 1 $= \sum_\lambda q^* t^* LLT_{\nu(\lambda)}(x;q)$

Combinatorial:

Closed form $=$ Infinite Series

Catalanimals

Classic: $\mathcal{F}(DH_n) = \sum_\lambda q^{\mathrm{dinv}(\lambda)} t^{\mathrm{area}(\lambda)} \mathcal{Y}_{\underline{\nu}(\lambda)}$

Goals I) Nonsymmetric Hall-Littlewood polynomials

a) Definition
b) Properties

II) LLT series

a) Algebraic construction from ns Hall-Littlewoods
b) Combinatorial connection

III) Cauchy formula

a) Statement
b) Some applications

Conventions

$X := X(GL_\ell) = \mathbb{Z}^\ell$ "weights"

$W = \mathfrak{S}_\ell =$ symmetric group on ℓ letters

$w \in \mathfrak{S} \longleftrightarrow (w(1), \ldots, w(\ell))$

$W \curvearrowright X$ via $w.\lambda = (\lambda_{w^{-1}(1)}, \ldots, \lambda_{w^{-1}(\ell)})$

$X_+ = \{ \lambda \in \mathbb{Z}^\ell \mid \lambda_1 \geq \cdots \geq \lambda_\ell \}$ "dominant weights"

For $\lambda \in X$, $\lambda_+ =$ dominant rearrangement of λ.

$=$ unique element in $S_\ell \lambda \cap X_+$

Eg $\lambda = (1,3,1,2)$, $\lambda_+ = (3,2,1,1)$,

$w(\lambda_+) = \lambda$ for $w = (3,1,4,2)$.

Hecke Algebra (of type A)

Recall $\mathfrak{S}_n = \left\langle s_1, \dots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i^2 = \mathrm{id} & i = 1, \dots, n-1, \\ s_i s_j = s_j s_i & |i-j| > 1, \text{ and} \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & i = 1, \dots, n-2. \end{array} \right\rangle$

$s_i = (i \leftrightarrow i+1)$

Def Hecke algebra $\mathcal{H} := \mathcal{H}(\mathfrak{S}_n) = \Bbbk_{q}$-algebra

generated by

$$\left\langle T_1, \dots, T_{n-1} \;\middle|\; \begin{array}{ll} (T_i - q)(T_i + 1) = 0 & i = 1, \dots, n-1 \\ T_i T_j = T_j T_i & |i-j| > 1, \text{ and} \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} & i = 1, \dots, n-2 \end{array} \right\rangle$$

Note
- $T_i^2 + (1-q)T_i - q = 0 \Rightarrow T_i^{-1} = q^{-1}(T_i + (1-q))$
- $q \longmapsto 1$ recovers $\Bbbk\mathfrak{S}_n$, the group algebra of $\mathfrak{S}_n$.

Def For $w \in \mathfrak{S}_n$, $w = s_{i_1} \cdots s_{i_\ell}$ with ℓ minimal is a reduced expression of w.

$\ell(w) :=$ length of a reduced expression for w

$$= \left| \{ i<j \mid w(i) > w(j) \} \right| = \mathrm{inv}(w)$$

Eg $w = s_1 s_2 s_1 = s_2 s_1 s_2 = (3,2,1)$ has $\ell(w) = 3$.

Def $T_w := T_{i_1} \cdots T_{i_\ell}$ for any reduced expression $w = s_{i_1} \cdots s_{i_\ell}$.

Rmk T_w is well-defined.

Prop $\{T_w\}_{w \in \mathfrak{S}_n}$ forms a k-basis of $\mathcal{H}$.

Demazure–Lusztig Operators

$\mathcal{H} \curvearrowright \Bbbk[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}] \cong \Bbbk X$ via Demazure–Lusztig operators

$$T_i f = q\, s_i f + (1-q) \frac{1}{1 - x_{i+1}/x_i} (s_i f - f)$$

for $(s_i f)(x_1, \dots, x_n) = f(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_\ell)$.

Note
- $s_i f - f$ antisymmetric in x_i, x_{i+1}

  $\Rightarrow s_i f - f$ divisible by $x_i - x_{i+1}$.
- $q \mapsto 1$ gives $T_i \longmapsto s_i$.

$$T_i f = q s_i f + (1-q)\frac{1}{1-x_{i+1}/x_i}(s_i f - f)$$

Eg $T_1(x_1 x_2) = q x_1 x_2$

$$T_1(x_1^2) = q x_2^2 + \underbrace{(1-q)\frac{1}{1-x_2/x_1}(x_2^2 - x_1^2)}_{(1-q)x_1(-x_1-x_2)}$$

$$= q x_2^2 + (q-1)x_1 x_2 + (q-1)x_1^2$$

$$T_1(x_2^2) = q x_1^2 + (1-q)\frac{1}{1-x_2/x_1}(x_1^2 - x_2^2)$$

$$= x_1^2 + (1-q)x_1 x_2$$

Nonsymmetric Hall-Littlewood polynomials

"Demazure-Lusztig on dominant weight monomial"

Def For $\lambda \in \mathbb{Z}^\ell$,

$$E_\lambda(x_1,\dots,x_\ell;q) := q^{-\ell(w)} T_w\, x^{\lambda_+} \qquad \left(x^\mu := x_1^{\mu_1}\cdots x_\ell^{\mu_\ell}\right)$$

for $w \in \mathfrak{S}_\ell$ such that $w(\lambda_+) = \lambda$.

Eg $\lambda = (2,0)$ $\quad E_{20}(x_1,x_2;q) = x_1^2$

$\lambda = (0,2)$ $\quad E_{02}(x_1,x_2;q) = q^{-1}\, T_1(x_1^2)$

$$= x_2^2 + (1-q^{-1})x_1x_2 + (1-q^{-1})x_1^2$$

Rmk $E_\lambda(x;q) = E_\lambda(x;0,q^{-1})$ for ns Macdonald poly $E_\lambda(x;q,t)$.

Bruhat order on $S_\ell \lambda_+$

Transitive closure of $s_i\lambda > \lambda$ if $\lambda_i - \lambda_{i+1} > 0$.

Eg

$S_3 \cdot (2,1,0)$

$$\begin{tikzcd}[column sep=small]
 & (0,1,2) \arrow[dl, dash] \arrow[dr, dash] & \\
(1,0,2) \arrow[d, dash] \arrow[drr, dash] & & (0,2,1) \arrow[d, dash] \arrow[dll, dash] \\
(1,2,0) \arrow[dr, dash] & & (2,0,1) \arrow[dl, dash] \\
 & (2,1,0) &
\end{tikzcd}$$

$S_3 \cdot (1,1,0)$

$$\begin{tikzcd}
(0,1,1) \arrow[d, dash] \\
(1,0,1) \arrow[d, dash] \\
(1,1,0)
\end{tikzcd}$$

Triangularity

$$E_\lambda(x;q) = x^\lambda + \sum_{\mu < \lambda} c_\mu x^\mu, \quad c_\mu \in \mathbb{Z}[q^{-1}]$$

for $\mu \le \lambda \iff \mu_+ < \lambda_+$ in dominance order or

$\mu_+ = \lambda_+$ and $\lambda \le \mu$ in $S_\ell \lambda_+$.

Eg $E_{102}(x;q) = \overset{x_1^1x_2^0x_3^2}{\overset{\parallel}{x^{102}}} + (1-q^{-1})(x^{201} + x^{120} + x^{111}) + (1-q^{-1})^2 x^{210}$

Inversions

$\mu \in \mathbb{R}^\ell$, $\mathrm{Inv}(\mu) := \{ i<j \mid \mu_i > \mu_j \}$

$\rightsquigarrow \sigma \in \mathfrak{S}_\ell$, $\mathrm{Inv}(\sigma) = \{ i<j \mid \sigma(i) > \sigma(j) \}$

For ε small & $\rho = (\ell-1, \dots, 1, 0)$, $\mathrm{Inv}(\mu + \varepsilon\rho) = \{ i<j \mid \mu_i \geq \mu_j \}$.

Eg $\mathrm{Inv}(212) = \{(1,2)\}$, $\mathrm{Inv}(212 + \varepsilon\rho) = \{(1,2), (1,3)\}$

Twisted nonsymmetric Hall-Littlewood polynomials for $\sigma \in \mathfrak{S}_\ell$

$$E_\lambda^\sigma(x;q) := q^{|\mathrm{Inv}(\sigma^{-1}) \cap \mathrm{Inv}(\lambda + \varepsilon\rho)|} T_{\sigma^{-1}}^{-1} E_{\sigma^{-1}(\lambda)}(x;q)$$

$$F_\lambda^\sigma(x;q) := E_{-\lambda}^{\sigma w_0}(x_1^{-1}, \dots, x_\ell^{-1}; q^{-1})$$

for $w_0 \in \mathfrak{S}_\ell$ longest word given by $w_0 = (\ell, \ell-1, \dots, 2, 1)$

Prop $E^{\sigma}_{\lambda}(x;q) = x^{\lambda} + \sum_{\mu < \lambda} c_{\mu} x^{\mu} \quad \forall \sigma \in \mathfrak{S}_{\ell}.$

Cor $\{E^{\sigma}_{\lambda}(x;q)\}_{\lambda \in \mathbb{Z}^{\ell}}$ and $\{F^{\sigma}_{\lambda}(x;q)\}_{\lambda \in \mathbb{Z}^{\ell}}$ are both bases for $k[x_1^{\pm 1}, \ldots, x_{\ell}^{\pm 1}]$ $\forall \sigma \in \mathfrak{S}_{\ell}$.

Orthogonality

Take constant term

$$\langle f, g \rangle_q := \langle x^0 \rangle\, f\, \bar{g} \prod_{1 \le i < j \le \ell} \underbrace{\frac{1 - x_i/x_j}{1 - q^{-1} x_i / x_j}}$$

Expand as geometric series

$$\bar{\cdot} : k[x_1^{\pm 1}, \ldots, x_{\ell}^{\pm 1}] \longrightarrow k[x_1^{\pm 1}, \ldots, x_{\ell}^{\pm 1}]$$
$$x_i \longmapsto x_i^{-1}$$
$$q \longmapsto q^{-1}$$

Prop $\langle E^{\sigma}_{\lambda}, \overline{F^{\sigma}_{\mu}} \rangle_q = \delta_{\lambda\mu} \quad \forall \lambda, \mu \in \mathbb{Z}^{\ell},\ \forall \sigma \in \mathfrak{S}_{\ell}.$

Eg

$$E_{02} = x_2^2 + (1-q^{-1})x_1x_2 + (1-q^{-1})x_1^2 \qquad F_{02} = x_2^2$$

$$E_{11} = x_1x_2 \qquad F_{11} = x_1x_2$$

$$E_{20} = x_1^2 \qquad F_{20} = x_1^2 + (1-q)x_1x_2$$

$$E_{02}^{w_0} = x_2^2 + (1-q^{-1})x_1x_2 \qquad F_{02}^{w_0} = x_2^2$$

$$E_{11}^{w_0} = x_1x_2 \qquad F_{11}^{w_0} = x_1x_2$$

$$E_{20}^{w_0} = x_1^2 \qquad F_{20}^{w_0} = x_1^2 + (1-q)x_1x_2 + (1-q)x_2^2$$

Recursions

Eg $$E^{213}_{101}(x;q) = x_1x_3 + (1-q^{-1})x_1x_2$$

$$= x_1\left(x_3 + (1-q^{-1})x_2\right)$$

$$= E_1(x_1)\, E^{12}_{01}(x_2,x_3)$$

In general $$E^{\sigma}_{\underbrace{\lambda_1,\dots,\lambda_k}_{\lambda_i},\ \underbrace{\lambda_{k+1},\dots,\lambda_\ell}_{\lambda_j}}(x;q) = E^{\sigma_1}_{\lambda_1,\dots,\lambda_k}(x;q)\, E^{\sigma_2}_{\lambda_{k+1},\dots,\lambda_\ell}(x;q)$$

$$\lambda_i \geq \lambda_j$$

Special Cases

$(\ell,\ell-1,\dots,2,1)$ $\qquad$ $(\ell-1,\dots,2,1)$

If $\lambda_\ell \leq \lambda_i\ \forall i$, $$E^{w_0}_{\lambda}(x_1,\dots,x_\ell;q) = E^{\hat{w}_0}_{\lambda_1,\dots,\lambda_{\ell-1}}(x_1,\dots,x_{\ell-1};q)\, x_\ell^{\lambda_\ell}$$

If $\lambda_1 \leq \lambda_i\ \forall i$, $$F^{w_0}_{\lambda}(x_1,\dots,x_\ell;q) = x_1^{\lambda_1}\, F^{\hat{w}_0}_{\lambda_2,\dots,\lambda_\ell}(x_2,\dots,x_\ell;q)$$

GL_ℓ - Characters

Weyl Symmetrization $\quad \sigma : k[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}] \to k[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]^{\mathfrak{S}_\ell}$

$$\sigma(f(x)) = \sum_{w \in \mathfrak{S}_\ell} w\left(\frac{f(x)}{\prod_{1 \le i < j \le \ell} (1 - x_j/x_i)} \right)$$

Irreducible characters

For $\lambda \in X_+$, $\chi_\lambda(x_1, \dots, x_\ell) = \sigma(x^\lambda)$.

If $\underbrace{\lambda_\ell \geq 0}_{\text{"polynomial characters"}}$, $\chi_\lambda(x_1, \dots, x_\ell) = s_\lambda(x_1, \dots, x_\ell) \leftarrow$ Schur polynomial

Note $\quad (x_1 \cdots x_\ell)^d \chi_\lambda = \chi_{\lambda + (d, \dots, d)} \quad \forall d \in \mathbb{Z}$

$$\sigma(f(x)) = \sum_{w \in \mathfrak{S}_\ell} w\left(\frac{f(x)}{\prod_{1 \le i < j \le \ell} (1 - x_j/x_i)}\right)$$

Eg $\chi_{20}(x_1, x_2) = \sigma(x_1^2)$

$$= \frac{x_1^2}{1 - x_2/x_1} + \frac{x_2^2}{1 - x_1/x_2}$$

$$= \frac{x_1^3 - x_2^3}{x_1 - x_2}$$

$$= x_1^2 + x_1 x_2 + x_2^2$$

$$\chi_{1\,-1}(x_1, x_2) = (x_1 x_2)^{-1} \chi_{20}(x_1, x_2)$$

$$= x_1 x_2^{-1} + 1 + x_1^{-1} x_2$$

LLT Series

For $\alpha, \beta \in \mathbb{Z}^\ell$, $\sigma \in \mathfrak{S}_\ell$,

$$\mathcal{L}^{\sigma}_{\beta/\alpha}(x;q) = \sigma\left(\frac{w_0\left(F^{\sigma^{-1}}_\beta(x;q)\,\overline{E^{\sigma^{-1}}_\alpha(x;q)}\right)}{\prod_{1\le i<j\le \ell}\left(1-q\,x_i/x_j\right)}\right)$$

geometric series

Eg $\mathcal{L}^{id}_{21/00} = \sigma\left(\dfrac{w_0\left(x_1^2x_2\cdot 1\right)}{1-q\,x_1/x_2}\right)$

$$= \sigma\left(x_1x_2^2 + q\,x_1^2x_2 + q^2x_1^3 + q^3x_1^4x_2^{-1} + \cdots\right)$$

$$= 0 + q\,\chi_{21} + q^2\chi_{30} + q^3\chi_{4\,-1} + \cdots$$

Polynomial part

$$\mathrm{pol}_X\left(\mathcal{X}_\lambda(x_1,\dots,x_\ell)\right) = \begin{cases} s_\lambda & \text{if } \lambda_\ell \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Rmk Not the same as killing monomials with negative exponents!

Eg $\mathrm{pol}_X(\underbrace{x_1 x_2^{-1} + x_1^{-1} x_2}_{\mathcal{X}_{1-1} - \mathcal{X}_{00}}) = -\mathcal{X}_{00} = -1$

Eg $\mathcal{L}^{id}_{21/00} = q\mathcal{X}_{21} + q^2 \mathcal{X}_{30} + q^3 \mathcal{X}_{4-1} + \cdots$

$\mathrm{pol}_X(\mathcal{L}^{id}_{21/00}) = q s_{21} + q^2 s_{30}$

Relationship between Series LLTs and LLT Polynomials

$$\mathrm{pol}_X \, \mathcal{L}^{\sigma}_{\beta/\alpha}(x;q) = \begin{cases} q^{h_\sigma(\beta/\alpha)} \, \mathcal{G}_{\sigma(\beta/\alpha)}(x;q^{-1}) & \text{if } \beta_i \geq \alpha_i \\ 0 & \text{otherwise} \end{cases}$$

Tuple of offset rows

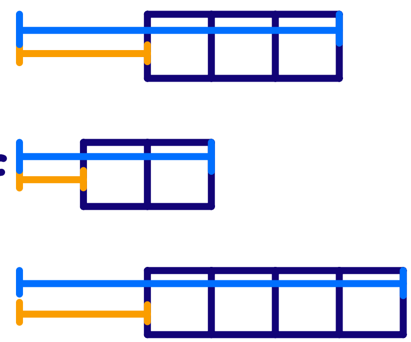
for $h_\sigma(\beta/\alpha)$ = "# σ-triples in β/α"

Eg

$\beta = 6, 3, 5$

$\alpha = 2, 1, 2$

$\beta/\alpha =$



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Perspectives on Catalanimals part I (b)

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Reminder of yesterday

(Twisted) Nonsymmetric Hall-Littlewood polynomials

$\forall \sigma \in \mathfrak{S}_\ell$, bases $\{E^\sigma_\lambda(x;q)\}_{\lambda\in\mathbb{Z}^\ell}$ and $\{F^\sigma_\lambda(x;q)\}_{\lambda\in\mathbb{Z}^\ell}$

which satisfy $\langle E^\sigma_\lambda, \overline{F^\sigma_\mu} \rangle_q = \delta_{\lambda\mu}$.

Series LLTs For $\alpha,\beta \in \mathbb{Z}^\ell$, $\sigma \in \mathfrak{S}_\ell$,

$$\mathcal{L}^\sigma_{\beta/\alpha}(x;q) = \underline{\sigma}\left(\frac{w_0\left(F^{\sigma^{-1}}_\beta(x;q)\,\overline{E^{\sigma^{-1}}_\alpha(x;q)}\right)}{\prod_{1\le i<j\le \ell}(1-q\,x_i/x_j)} \right)$$

Weyl symmetrization

geometric series

$$= \underline{H}_q\left(w_0\left(F^{\sigma^{-1}}_\beta(x;q)\,\overline{E^{\sigma^{-1}}_\alpha(x;q)}\right)\right) \text{ for } \underline{H}_q(f) := \underline{\sigma}\left(\frac{f}{\prod_{i<j}(1-q\,x_i/x_j)}\right)$$

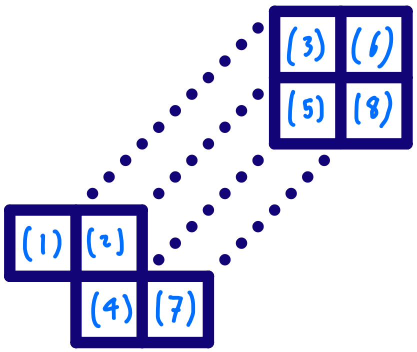
LLT Polynomials

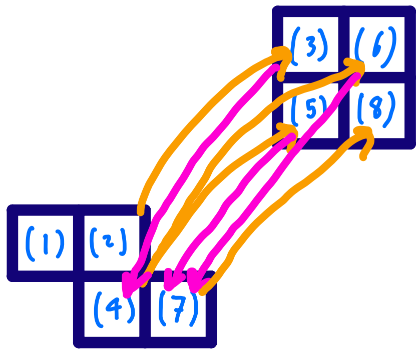
$\underline{\nu} = (\nu^{(1)}, \dots, \nu^{(k)})$

Skew partitions

$$\mathcal{G}_{\underline{\nu}}(X;q) = \sum_{T \in SSYT(\underline{\nu})} q^{inv(T)} x^{T}$$

Eg $\underline{\nu} = ((32)\backslash(1), (33)\backslash(11))$



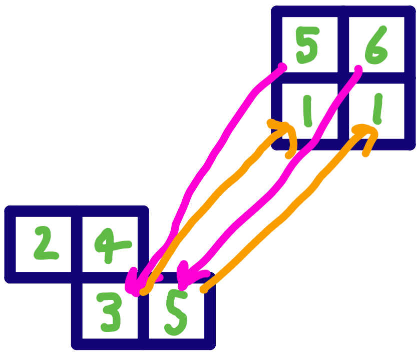


Attacking Pairs

(2),(3)
(4),(5)
(4),(6)
(7),(8)

(3),(4)
(5),(7)
(6),(7)

T



$inv(T) = 4$

$x^T = x_1^2 x_2 x_3 x_4 x_5 x_6$

Relationship between Series LLTs and LLT Polynomials

$$\mathrm{pol}_X \mathcal{L}^{\sigma}_{\beta/\alpha}(x;q) = \begin{cases} q^{h_\sigma(\beta/\alpha)} \mathcal{G}_{\sigma(\beta/\alpha)}(x;q^{-1}) & \text{if } \beta_i \geq \alpha_i \\ 0 & \text{otherwise} \end{cases}$$

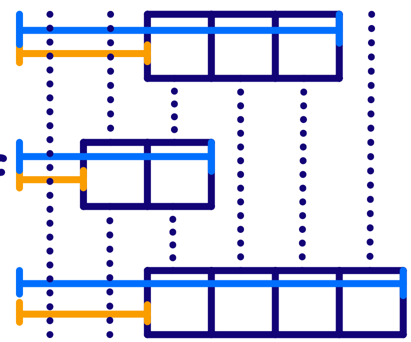
Tuple of offset rows

for $h_\sigma(\beta/\alpha) =$ "# σ-triples in β/α"

Eg

$\beta = 6,3,5$

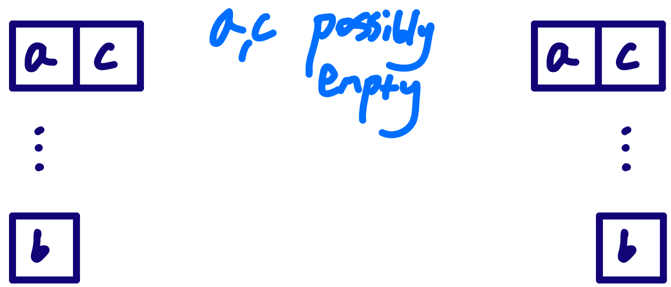
$\alpha = 2,1,2$

$\beta/\alpha =$ 

σ-triples of boxes (a,b,c) in β/α

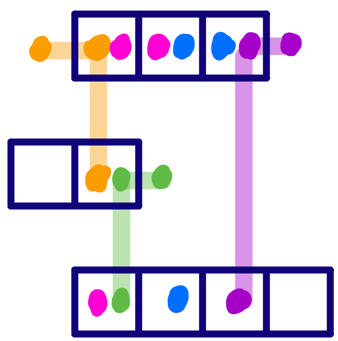
row j

row i



if $\sigma(j) > \sigma(i)$ if $\sigma(j) < \sigma(i)$

Eg $\sigma = 312$ β/α



$\sigma(3) = 2$ row 3

$\sigma(2) = 1$ row 2

$\sigma(1) = 3$ row 1

$\rightsquigarrow h_{312}(\beta/\alpha) = 5$

Cauchy Identity

$$\frac{\prod_{1\le i<j\le \ell}(1-qtx_iy_j)}{\prod_{1\le i\le j\le \ell}(1-tx_iy_j)} = \sum_{\underline{a}\in\mathbb{N}^\ell} t^{|\underline{a}|} E^{\sigma}_{\underline{a}}(x;q^{-1}) F^{\sigma}_{\underline{a}}(y;q)$$

$$\forall\, \sigma \in \mathfrak{S}_\ell$$

Proof via "Manipulatorics" with $\langle \cdot, \cdot \rangle_q$ and duality of E's and F's.

Later, we will see expressions like

Cauchy-like factor

$$H_\lambda := \underline{\sigma}\left(\frac{x^\lambda \prod_{i+1<j}(1-qt\, x_i/x_j)}{\prod_{i<j}(1-q\, x_i/x_j)\prod_{i<j}(1-t\, x_i/x_j)} \right)$$

$$= \underline{H}_q\left(x^\lambda \frac{\prod_{i+1<j}(1-qt\, x_i/x_j)}{\prod_{i<j}(1-t\, x_i/x_j)} \right)$$

and want to expand into LLTs.

Special Case $\lambda = (1, \dots, 1)$ (so $H_\lambda \leftrightarrow \nabla e_\ell$)

$\rightsquigarrow x^\lambda = x_1 x_2 \cdots x_\ell$

$(\ell-1,\dots,2,1)\in S_{\ell-1}$

Cauchy formula in $\ell-1$ variables with $x_i \longmapsto x_i^{-1}$, $\sigma=\hat{w}_0$

$y_j \longmapsto x_{j+1}$

$$\frac{\prod_{i+1<j}(1-qt\,x_j/x_i)}{\prod_{i<j}(1-t\,x_j/x_i)} = \sum_{\underline{a}} t^{|\underline{a}|} F^{\hat{w}_0}_{(a_{\ell-1},\dots,a_1)}(x_2,\dots,x_\ell;q)\,\overline{E^{\hat{w}_0}_{(a_{\ell-1},\dots,a_1)}(x_1,\dots,x_{\ell-1};q)}$$

$$\Big\downarrow\ {\scriptstyle *\, x_1\cdots x_\ell}$$

$$\sum_{\underline{a}} t^{|\underline{a}|}\underbrace{x_1\,F^{\hat{w}_0}_{(a_{\ell-1}+1,\dots,a_1+1)}(x_2,\dots,x_\ell;q)}\;\underbrace{\overline{E^{\hat{w}_0}_{(a_{\ell-1},\dots,a_1)}(x_1,\dots,x_{\ell-1};q)}}$$

Recursive Identities: $F^{w_0}_{(1,a_{\ell-1}+1,\dots,a_1+1)}(x_1,\dots,x_\ell;q)$ $\quad\overline{E^{w_0}_{(a_{\ell-1},\dots,a_1,0)}(x_1,\dots,x_\ell;q)}$

Applying $\underline{H}_q(w_0(-))$ to both sides gives...

Stable Shuffle Theorem (Special case)

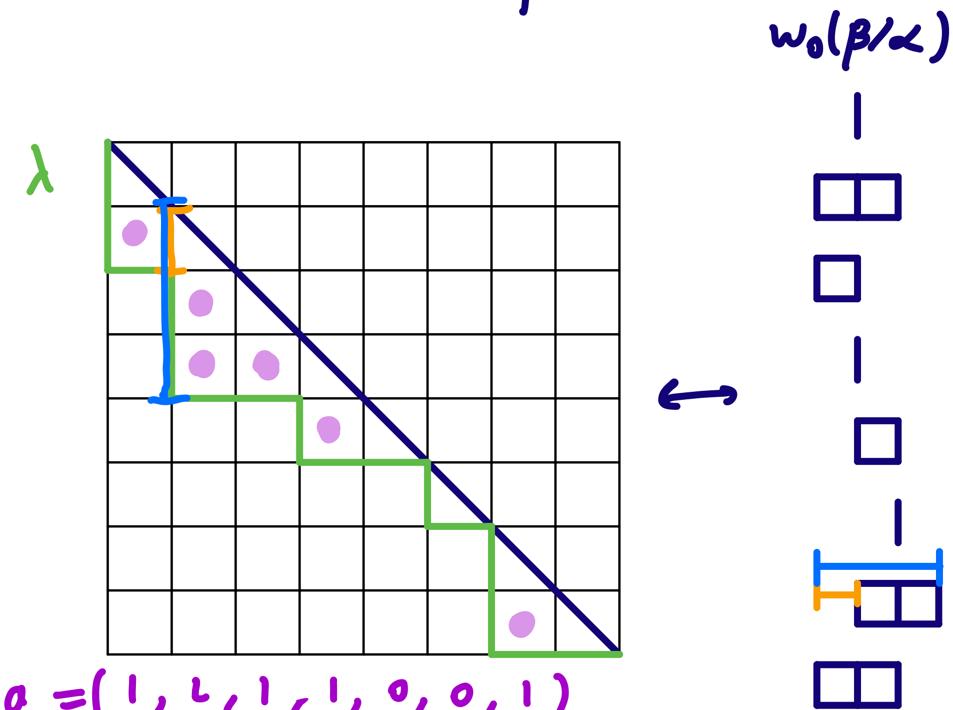
$$H_{(1^\ell)} = \sum_{\underline{a} \in \mathbb{N}^{\ell-1}} t^{|a|} \mathcal{L}^{w_0}_{\underbrace{(1^\ell)+(0;a_{\ell-1},\dots,a_1)}_{\beta} / \underbrace{(a_{\ell-1},\dots,a_1;0)}_{\alpha}}(X;q)$$

$$\Big\downarrow \ \mathrm{pol}_X(-)$$

$$\mathrm{pol}_X(H_{(1^\ell)}) = \sum_{\substack{\underline{a} \in \mathbb{N}^{\ell-1} \\ a_{i-1} \le 1+a_i \\ a_{\ell-1} \le 1}} t^{|a|} q^{h_{w_0}(\beta/\alpha)} \mathcal{G}_{w_0(\beta/\alpha)}(X;q^{-1})$$

Connecting to Dyck Paths

$\underline{a} \longleftrightarrow$ Column area sequence



$\underline{a} = (1, 2, 1, 1, 0, 0, 1)$

Rmk $\quad h_{w_0}(\beta/\alpha) = \mathrm{dinv}(\lambda)$ via bijective argument

Conclusion

Cauchy identity + manipulating ns Hall-Littlewood polynomials & LLT polynomials gives

$$H_\lambda = \sum t^{|a|} \mathcal{L}^{\sigma}_{w_0(\lambda)+(0;\underline{a})/(\underline{a};0)}(x;q)$$

for various λ.

More generally, upgrading these techniques allows for more interesting LLT expansions of "Schur Catalanimals."